Direct data-driven control of constrained linear systems

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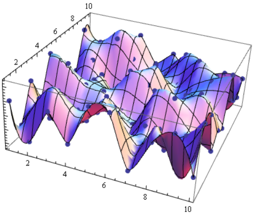

ERNSI Workshop @Lyon, France, September 25, 2017

Joint work with R. Tóth, D. Piga, A. Bemporad and S.M. Savaresi

Outline

- Problem formulation
- Direct design of LPV controllers from data
- Constraint management and performance boost
- Concluding remarks

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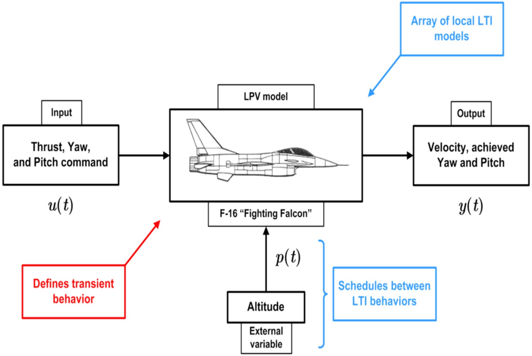

- The dynamic relationship between $u(t)$ and $y(t)$
  - ✓ is **linear**
  - ✓ depends on a **measurable** signal, the so-called scheduling variable $p(t)$

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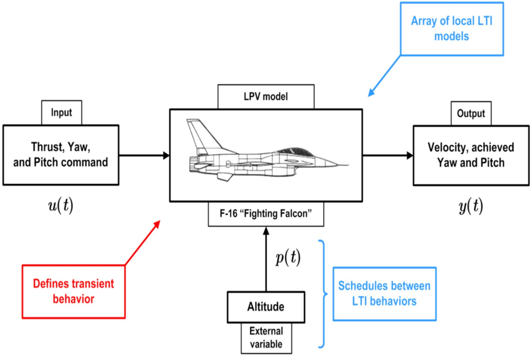

- The dynamic relationship between $u(t)$ and $y(t)$
  - ✓ is **linear**
  - ✓ depends on a **measurable** signal, the so-called scheduling variable $p(t)$
- **Linear** robust and gain-scheduled control solution can be employed for effective control of complex nonlinear systems!

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- Standard approach: system identification + model-based control

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- Identification (global approaches)
  - ✓ **State-space (SS) models**: (Nemani et al, 1995), (Lee and Poolla, 1997 & 1999) (Lovera et al., 1998), (Sznaier and Mazzaro, 2001 & 2003), (Felici et al., 2007), (van Wingerden and Verhaegen, 2008), (Rizvi et al., 2017)
  - ✓ **Input/output (IO) models**: (Bamieh and Giarré, 1999 & 2002), (Previdi and Lovera, 2003 & 2004), (Toth et al., 2007 & 2008), (Piga et al., 2015)

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- Model-based control
  - ✓ **SS models**: many approaches since J. Shamma's thesis in 1988
  - ✓ **IO models**: some recent studies, e.g., (Ali et al., 2010), (Cerone et al., 2012), (Wollnack et al., 2013), (Abbas et al., 2016 & 2017)

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- Problems:
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  - ✓ **IO control** design is mainly based on gradient-based BMI solvers and generally suffers from high computational cost
  - ✓ Assessment of performance is often done in continuous-time but **discretization** of LPV models is still an open problem (Toth et al., 2008)

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  - ✓ The way the **modeling errors** affect the control performance is unknown for most design methods
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- **Idea: direct identification of the discrete-time controller in its IO representation** ("from data to implementation")

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- Many possible ways to address the problem. Here:
  - ✓ Stochastic framework
  - ✓ Model reference control formulation

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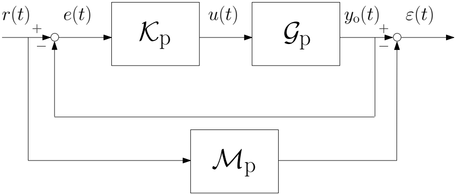

$$
\begin{aligned}
\theta_o = \arg\min_{\theta,\varepsilon} \|\varepsilon\|^2 & \\
\text{s.t.} \quad \varepsilon(t) &= y_o(t) - M(p,t,q^{-1})r(t) \\
A(p,t,q^{-1})y_o(t) &= B(p,t,q^{-1})u(t) \\
A_K(p,t,q^{-1},\theta)u(t) &= B_K(p,t,q^{-1},\theta)(r(t) - y_o(t))
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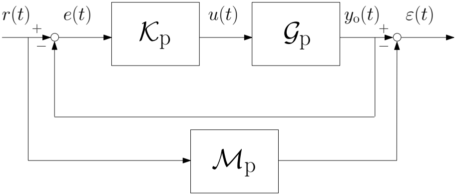

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Model-dependent! (unique model-based contribution: Abdullah and Zribi, 2009)

Assumptions

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A2. $\mathcal{G}$ is stable.

Definition D1. An LPV system $\mathcal{G}$ is **stable** if, for all trajectories $\{u(t), y(t), p(t)\}$ satisfying $\mathcal{G}$ with $u(t) = 0,\ t \geq 0$, $\exists\ \delta > 0$ s.t. $|y(t)| \leq \delta, \forall t \geq 0$.

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Definition D2. Given a causal LPV map $\mathcal{M}$ with input r, scheduling signal p and output y, the causal LPV mapping $M^{\dagger}$ that gives r as output when fed by y, for any trajectory of p, is called the **left inverse** of $\mathcal{M}$.

The main idea

- Problem reformulation:

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  - Use of IO data instead of constraint 2

- N finite and $y(t) = y_o(t) + w(t)$, with $w(t)$ any zero mean stationary noise

The data-driven optimization problem

- Final control design problem:

$$
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&\min_{\theta,\varepsilon} \|\varepsilon\|^2 \quad \text{s.t.} \\
A_K(p,\theta)u(t) \quad &= \quad B_K(p,\theta)(M^\dagger(p)\varepsilon(t) + M^\dagger(p)y(t) - y(t)) \\
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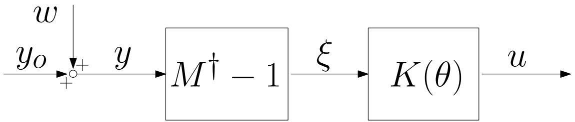
is an **identification problem**



- ✓ ξ is the (virtual) noisy input
- ✓ u is the noiseless output
- ✓ ε is the residual
- ✓ $K(\theta)$ is the system to identify

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$$\min_{\theta,\varepsilon} \|\varepsilon\|^2 \quad \text{s.t.}$$
$$A_K(p,\theta)u(t) = B_K(p,\theta)(M^\dagger(p)\varepsilon(t) + \xi(t))$$
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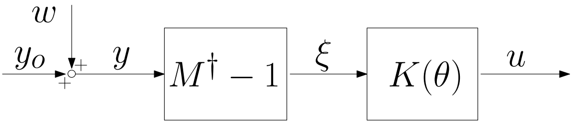

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- Critical points:
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- Purely data-driven optimization problem with N constraints
- Critical points:
  - ✓ the original and the current problems are equivalent when $y(t)$ is noisy?
  - ✓ problem complexity depends on the controller parameterization
  - ✓ computation of $M^{\dagger}(p)$ from $\mathcal{M}$ defined as

$$
\begin{aligned}
x_M(t+1) &= A_M(p,t)x_M(t) + B_M(p,t)r(t) \\
y(t) &= C_M(p,t)x_M(t) + D_M(p,t)r(t)
\end{aligned}
$$

Computation of $M^{\dagger}(p)$

- Analytical solution:

Proposition

Assume that $D_M(p,t) \neq 0, \forall p$ such that $\exists\ D_M^{-1}(p,t)$ with $D_M^{-1}(p,t) D_M(p,t) = 1,\ \forall p$. Define the state-space representation of $M^{\dagger}$ as

$$
\begin{aligned}
x_{M^\dagger}(t+1) &= A_{M^\dagger}(p,t) x_{M^\dagger}(t) + B_{M^\dagger}(p,t) y(t) \\
r(t) &= C_{M^\dagger}(p,t) x_{M^\dagger}(t) + D_{M^\dagger}(p,t) y(t)
\end{aligned}
$$

The matrices describing $M^{\dagger}$ can be computed as

$$
\begin{aligned}
A_{M^\dagger}(p,t) &= A_M(p,t) - B_M(p,t) D_M^{-1}(p,t) C_M(p,t), \\
B_{M^\dagger}(p,t) &= B_M(p,t) D_M^{-1}(p,t), \\
C_{M^\dagger}(p,t) &= -D_M^{-1}(p,t) C_M(p,t), \\
D_{M^\dagger}(p,t) &= D_M^{-1}(p,t).
\end{aligned}
$$

Two different scenarios

- In real applications, we encounter two different scenarios

  ✓ CASE A: **Fixed-structure controller tuning** (namely, parameter optimization, e.g. gain-scheduled PID tuning with affine parameterization)

  ✓ CASE B: **LPV controller design** (both structure selection and parameter optimization are needed, e.g. gain-scheduled PID tuning with unknown parameterization)

Two different scenarios - cont'd

- Controller parameterization:

$$A_K(p,t,q^{-1},\theta)u(t) = B_K(p,t,q^{-1},\theta)(r(t)-y(t))$$

$$A_K(p,t,q^{-1}) = 1 + \sum_{i=1}^{n_{\mathrm{a}_K}} a_i^K(p,t)q^{-i}$$

$$B_K(p,t,q^{-1}) = \sum_{i=0}^{n_{\mathrm{b}_K}} b_i^K(p,t)q^{-i}$$

$$a_i^K(p,t) = \sum_{j=1}^{n_0} a_{i,j}^K f_{i,j}(p,t)$$

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- CASE A: $f_{i,j}(p, t)$ and $g_{i,j}(p, t)$ are **a-priori** defined nonlinear (possibly dynamic) functions of p.

Fixed-structure controller tuning

- CASE A:

$$\theta = [\underline{a}_1^\top \; \dots \; \underline{a}_{n_{\mathrm{a}_K}}^\top \; \underline{b}_0^\top \; \dots \; \underline{b}_{n_{\mathrm{b}_K}}^\top]^\top$$

$$\underline{a}_i = [a_{i,1}^K \; \dots \; a_{i,n_i}^K]^\top, \; \underline{b}_i = [b_{i,1}^K \; \dots \; b_{i,m_i}^K]^\top$$

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- The problem of interest

$$\begin{aligned} &\min_{\theta,\varepsilon} \|\varepsilon\|^2 \quad \text{s.t.} \\ A_K(p,\theta)u(t) \quad &= \quad B_K(p,\theta)(M^\dagger(p)\varepsilon(t) + M^\dagger(p)y(t) - y(t)), \forall t \end{aligned}$$

✓ is **convex** if $B_K(q^{-1}, p, \theta)$ is independent of θ

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✓ is **convex** if $B_K(q^{-1}, p, \theta)$ is independent of θ

✓ is **bi-convex** if $B_K(q^{-1}, p, \theta)$ is linear in θ

Fixed-structure controller tuning - cont'd

- Controller dynamics:

$$A_K(p,\theta)u(t) = B_K(p,\theta)\xi(t) + B_K(p,\theta)M^{\dagger}(p)\varepsilon(t)$$

Fixed-structure controller tuning - cont'd

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- Define:

$$\phi(\xi,t) = [-u(t-1)f_{1,0}(p,t),\ldots,\xi(t)g_{0,0}(p,t),\xi(t-n_{\mathrm{b}_K})g_{n_{\mathrm{b}_K},1}(p,t),\ldots]^{\top}$$

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- "Regression-like" form:

$$u(t) = \phi^{\top}(\xi,t)\theta + B_K(p,t,\theta)M^{\dagger}(p,t)\varepsilon(t)$$

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- θ-dependent residual (and noisy input) $\rightarrow$ instrumental variables
- "Least squares"-like solution $\hat{\theta}_{\mathrm{IV}}$

Proposition

$$\lim_{N\to\infty} \hat{\theta}_{\mathrm{IV}} = \theta_o, \textit{ w.p. 1.}$$

Illustrative example

- LPV system $\mathcal{G}$ (with $\mathbb{P} = [-0.4, 0.4]$)

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- Reference LPV behaviour $\mathcal{M}$

$$\begin{aligned} x_M(t+1) &= A_M(p,t)x_M(t) + B_M(p,t)r(t) \\ y_M(t) &= C_M(p,t)x_M(t) + D_M(p,t)r(t) \end{aligned}$$

$$A_M(p,t) = \begin{bmatrix} -1 & 1 \\ -1-\Delta p(t) & 1 \end{bmatrix}, \; B_M(p,t) = \begin{bmatrix} 1+p(t) \\ 1+\Delta p(t) \end{bmatrix},$$

$$C_M = [1 \;\; 0], \; D_M = 0, \Delta p(t) = p(t) - p(t-1)$$

Illustrative example

- LPV system $\mathcal{G}$ (with $\mathbb{P} = [-0.4, 0.4]$)

$$
\begin{array}{rcl}
x_G(t+1) & = & p(t)x_G(t) + u(t) \\
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$$

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$$

- Gain-scheduled PI controller $\mathcal{K}$

$$
\begin{array}{rcl}
x_K(t+1) & = & x_K(t) + \left(\theta_0(p,t) + \theta_1(p,t)\right)\left(r(t) - y(t)\right) \\
u(t) & = & x_K(t) + \theta_0(p,t)\left(r(t) - y(t)\right)
\end{array}
$$

$$
\theta_0(p,t) = \theta_{00} + \theta_{01}p(t), \; \theta_1(p,t) = \theta_{10} + \theta_{11}p(t-1)
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Illustrative example

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- Reference LPV behaviour $\mathcal{M}$ (achievable!)

$$\begin{array}{rcl} x_M(t+1) & = & A_M(p,t)x_M(t) + B_M(p,t)r(t) \\ y_M(t) & = & C_M(p,t)x_M(t) + D_M(p,t)r(t) \end{array}$$

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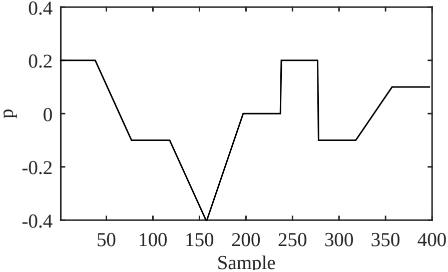
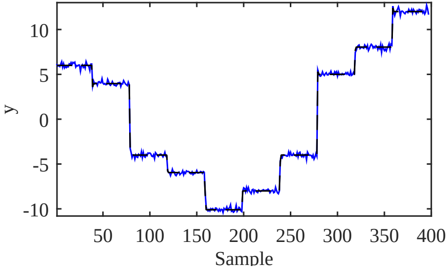
Illustrative example - cont'd





Mean tracking error: **less than 1%**

$$\theta_0^\circ(p,t) = 1 + p(t) \quad \rightarrow \quad \hat{\theta}_0(p,t) = 0.9852 + 1.0166p(t)$$
$$\theta_1^\circ(p,t) = -p(t-1) \quad \rightarrow \quad \hat{\theta}_1(p,t) = -0.0153 - 0.9860p(t-1)$$

Price to pay: two experiments plus a least squares estimation

Two different scenarios - cont'd

- Controller parameterization:

$$A_K(p,t,q^{-1},\theta)u(t) = B_K(p,t,q^{-1},\theta)(r(t)-y(t))$$

$$A_K(p,t,q^{-1}) = 1 + \sum_{i=1}^{n_{\mathrm{a}_K}} a_i^K(p,t)q^{-i}$$

$$B_K(p,t,q^{-1}) = \sum_{i=0}^{n_{\mathrm{b}_K}} b_i^K(p,t)q^{-i}$$

$$a_i^K(p,t) = \sum_{j=1}^{n_0} a_{i,j}^K f_{i,j}(p,t)$$

$$b_i^K(p,t) = \sum_{j=0}^{m_i} b_{i,j}^K g_{i,j}(p,t)$$

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- CASE B: $f_{i,j}(p,t)$ and $g_{i,j}(p,t)$ are **unknown** nonlinear (possibly dynamic) functions of p.

Nonparametric tuning

- CASE B:

$$a_i^K(p,t) = \theta_i^\top \psi_i(p,t) \quad i = 1,\ldots,n_{\mathrm{a}_K}$$
$$b_i^K(p,t) = \theta_{i+n_{\mathrm{a}_K}+1}^\top \psi_{i+n_{\mathrm{a}_K}+1}(p,t) \quad i = 0,\ldots,n_{\mathrm{b}_K}$$

- $\theta_i \in \mathbb{R}^{n_{\mathrm{H}}}$ is a vector of unknown parameters and $\psi_i(p,t)$ (with $i = 1,\ldots,n_{\mathrm{a}_K}+n_{\mathrm{b}_K}+1$) is a nonlinear map from $\mathbb{P}$ to an n_{H}-dimensional space, commonly referred to as the **feature space**

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- Neither ψ_i nor n_H are specified. Potentially, θ_i and $\psi_i(p,t)$ can be infinite-dimensional vectors!

Nonparametric tuning - cont'd

- Controller dynamics:

$$A_K(p,\theta)u(t) = B_K(p,\theta)\xi(t) + B_K(p,\theta)M^{\dagger}(p)\varepsilon(t)$$

Nonparametric tuning - cont'd

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- Define

$$x(\xi,t)=[-u(t-1)\ \ \ldots - u(t-n_{\mathrm{a}_K})\ \ \xi(t)\ \ \ldots\ \ \xi(t-n_{\mathrm{b}_K})]^{\top}.$$

Nonparametric tuning - cont'd

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- "Regression-like" form:

$$u(t) = \sum_{i=1}^{n_{\mathrm{f}}} \theta_i^{\top}\psi_i(p,t)x_i(\xi,t) + \underbrace{B_K(p,\theta,\psi_i)M^{\dagger}(p)\varepsilon(t)}_{\varepsilon_u(t)}$$

Nonparametric tuning - cont'd

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- θ- and ψ_i-dependent residual (and noisy input) $\rightarrow$ instrumental variables

Nonparametric tuning - cont'd

- Different problem formulation ("primal" form - still convex):

$$
\min_{\theta_i, \varepsilon_u} \frac{1}{2} \sum_{i=1}^{n_\mathrm{f}} \theta_i^\top \theta_i \; + \; \frac{\gamma}{2N^2} \sum_{i=1}^{n_\mathrm{f}} \left\| \sum_{t=1}^{N} z_i(t) \varepsilon_u(t) \right\|_2^2
$$

$$
\text{s.t. } \varepsilon_u(t) = u(t) - \sum_{i=1}^{n_\mathrm{f}} \theta_i^\top \psi_i(p, t) x_i(\xi, t), \quad \forall t \in \mathcal{I}_1^N
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Proposition

$$\lim_{N\to\infty} \hat{\theta}_{\mathrm{NP,IV}} = \theta^\circ - R^{-1}\gamma^{-1}\theta^\circ,$$

$$R = \lim_{N\to\infty} \gamma^{-1} I + \frac{1}{N^2}\Psi^\top Z Z^\top \Psi$$

Nonparametric tuning - cont'd

- Problem: $\hat{\theta}_{\mathrm{NP,IV}}$ cannot be computed since an explicit representation of $\psi_i(p, t)$ and $z_i(t)$ would be needed

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- Dual problem:

$$\mathcal{L}(\alpha,\theta,E) = \frac{1}{2}\sum_{i=1}^{n_{\mathrm{f}}}\theta_i^\top\theta_i \;+\; \frac{\gamma}{2N^2}\sum_{i=1}^{n_{\mathrm{f}}}\left\|\Psi_i^\top X_i(\hat{\xi})E\right\|_2^2 + \\ -\alpha^\top\left(E - U + \sum_{i=1}^{n_{\mathrm{f}}} X_i(\xi)\Psi_i\theta_i\right),$$

where $\alpha \in \mathbb{R}^N$ are **Lagrangian multipliers**

Nonparametric tuning - cont'd

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  where $\alpha \in \mathbb{R}^N$ are **Lagrangian multipliers**
- From *Karush-Kuhn-Tucker* (KKT) conditions for all $i = 1, \ldots, n_\mathrm{f}$:

$$\alpha = R_\mathrm{D}^{-1}(\Psi_i)\frac{1}{N^2}\sum_{i=1}^{n_\mathrm{f}} X_i(\hat{\xi})\Psi_i\Psi_i^\top X_i(\hat{\xi})U,$$

$$R_\mathrm{D}(\Psi_i) = \gamma^{-1} I + \frac{1}{N^2}\sum_{i=1}^{n_\mathrm{f}} X_i(\hat{\xi})\Psi_i\Psi_i^\top X_i(\hat{\xi}) \sum_{j=1}^{n_\mathrm{f}} X_j(\xi)\Psi_j\Psi_j^\top X_j(\xi).$$

Nonparametric tuning - cont'd

- Define the *Grammian* matrix as $\Omega_i = \Psi_i \Psi_i^\top$.
- According to the Mercer's theorem, the generic (t, k)-th entry of Ω_i can be described by a positive definite *kernel* function $\kappa_i(p, t, k)$, i.e.,

$$[\Omega_i]_{t,k} = \left\langle \psi_i(p, t), \psi_i(p, k) \right\rangle = \kappa_i(p, t, k).$$

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- In our case:

$$\alpha = R_{\mathrm{D}}^{-1}(\Omega_i) \frac{1}{N^2} \sum_{i=1}^{n_{\mathrm{f}}} X_i(\hat{\xi}) \Psi_i \Psi_i^\top X_i(\hat{\xi}) U$$

with

$$R_{\mathrm{D}}(\Omega_i) = \gamma^{-1} I + \frac{1}{N^2} \sum_{i=1}^{n_{\mathrm{f}}} X_i(\hat{\xi}) \Psi_i \Psi_i^\top X_i(\hat{\xi}) \sum_{j=1}^{n_{\mathrm{f}}} X_j(\xi) \Psi_i \Psi_i^\top X_j(\xi).$$

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- A typical choice of kernel is the **Radial Basis Function** (RBF) kernel:

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- Once the Lagrangian multipliers α are computed, the p-dependent coefficient functions $a_i^K(p,t)$ and $b_i^K(p,t)$ characterizing the LPV controller are obtained as

$$a_i^K(\cdot) = \psi_i^\top(\cdot)\theta_i = \psi_i^\top(\cdot)\Psi_i^\top X_i(\xi)\alpha = \sum_{t=1}^{N} \underbrace{\psi_i^\top(\cdot)\psi_i(p,t)}_{\kappa_i(p,t,\cdot)} x_i(\xi,t)\alpha_t,$$

$$b_i^K(\cdot) = \psi_{i+n_{a_K}+1}^\top(\cdot)\theta_{i+n_{a_K}+1} = \psi_{i+n_{a_K}+1}^\top(\cdot)\Psi_{i+n_{a_K}+1}^\top X_{i+n_{a_K}+1}(\xi)\alpha =$$

$$= \sum_{t=1}^{N} \underbrace{\psi_{i+n_{a_K}+1}^\top(\cdot)\psi_{i+n_{a_K}+1}(p,t)}_{\kappa_{i+n_{a_K}+1}(p,t,\cdot)} x_{i+n_{a_K}+1}(\xi,t)\alpha_t.$$

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$$\begin{aligned} u(t) = a_1^K(p(t), p(t-1))u(t-1) + \\ + b_0^K(p(t), p(t-1))\,(r(t) - y(t)) + \\ + b_1^K(p(t), p(t-1))\,(r(t-1) - y(t-1)) \end{aligned}$$

Illustrative example Reprise - cont'd

- Same IO data as before
- Now the dependence of a_1^K, b_0^K and b_1^K on $p(t)$ and $p(t-1)$ is **not** a-priori specified (neither the integral action is fixed)

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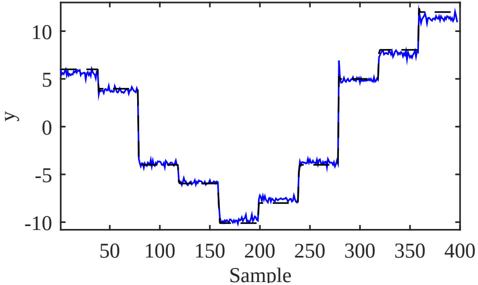

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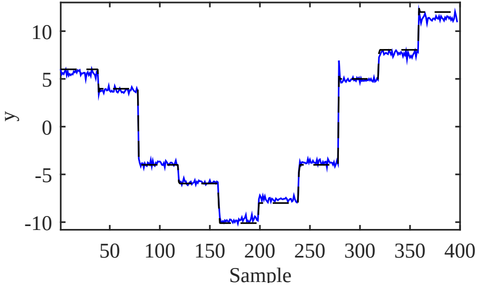

	parametric	**nonparametric**
MSE	0.0407	0.1565

A case study

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  - ✓ Assumption **A1** may not be verified ($\mathcal{M}$ is usually given as a - barely achievable - LTI model)
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- A more realistic case study :

Voltage-controlled DC motor with an unbalanced disk

A case study - cont'd

- Continuous-time system dynamics

$$\begin{bmatrix} \dot{\theta}(\tau) \\ \dot{\omega}(\tau) \\ \dot{I}(\tau) \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{b}{J} & \frac{K}{J} \\ 0 & -\frac{K}{L} & -\frac{R}{L} \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ \frac{mgl}{J} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{\sin(\theta(\tau))}{\theta(\tau)} \right) \begin{bmatrix} \theta(\tau) \\ \omega(\tau) \\ I(\tau) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} V(\tau)$$

$$y(\tau) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta(\tau) \\ \omega(\tau) \\ I(\tau) \end{bmatrix},$$

A case study - cont'd

- Continuous-time system dynamics

$$\begin{bmatrix} \dot{\theta}(\tau) \\ \dot{\omega}(\tau) \\ \dot{I}(\tau) \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{b}{J} & \frac{K}{J} \\ 0 & -\frac{K}{L} & -\frac{R}{L} \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ \frac{mgl}{J} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{\sin(\theta(\tau))}{\theta(\tau)} \right) \begin{bmatrix} \theta(\tau) \\ \omega(\tau) \\ I(\tau) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} V(\tau)$$

$$y(\tau) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta(\tau) \\ \omega(\tau) \\ I(\tau) \end{bmatrix},$$

- Third order system with dynamics depending on the load angular position θ

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- The system is **quasi LPV** (more realistic setting)

A case study - cont'd

- Data-driven controller tuning:
  - ✓ 1st order LTI reference model (≈ 0.15 Hz dynamics)

$$x_M(t+1) = 0.99x_M(t) + 0.01r(t)$$
$$\theta_M(t) = x_M(t)$$

  - ✓ 4th order nonparametric controller with integral action

$$u(t) = \sum_{i=1}^{4} a_i^K(\Pi(t))u(t-i) + \sum_{j=0}^{4} b_j^K(\Pi(t))e_{int}(t-j)$$
$$e_{int}(t) = e_{int}(t-1) + (r(t) - y(t)),$$
$$\Pi(t) = [p(t-1)\ p(t-2)\ p(t-3)\ p(t-4)]^\top,$$

A case study - cont'd

- γ and $\sigma_i, \ \forall i$ are chosen through cross-validation

A case study - cont'd

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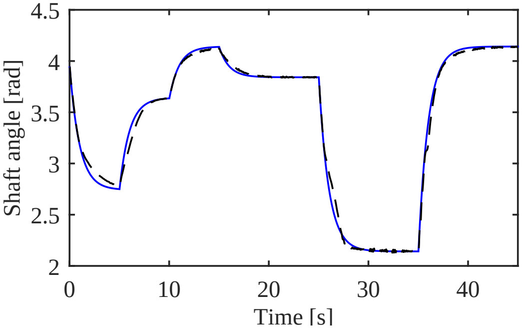
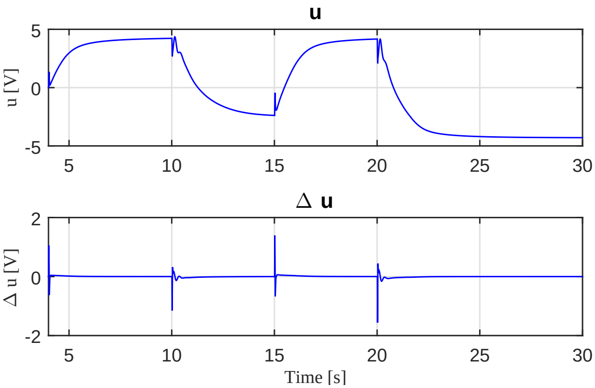


Closed-loop matching satisfactory almost everywhere (...but is the reference model acceptable?)

Moreover...

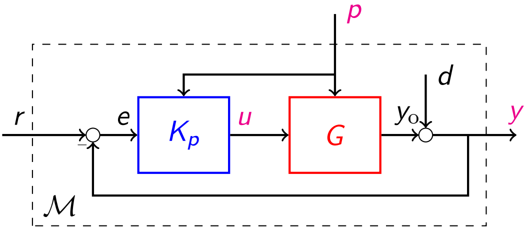


Constraints on the input signals cannot be handled

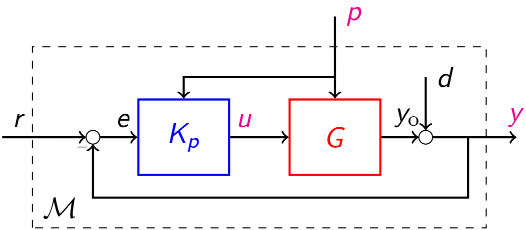
Outline

- Problem formulation
- Direct design of LPV controllers from data
- Constraint management and performance boost
- Concluding remarks

Hierarchical control architecture

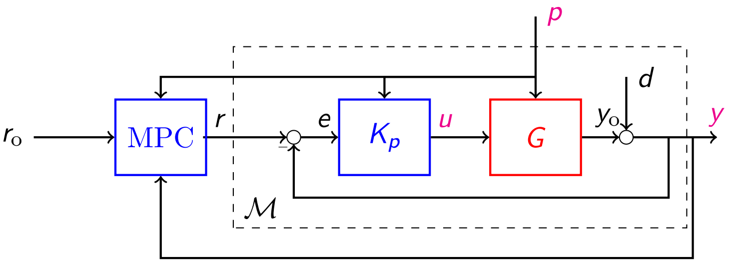


Hierarchical control architecture



The model $\mathcal{M}$ describes the relation between r and y!

Hierarchical control architecture



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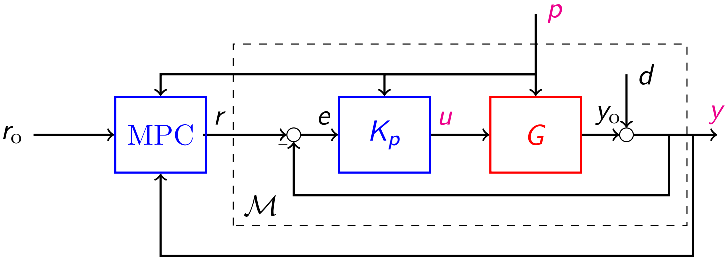

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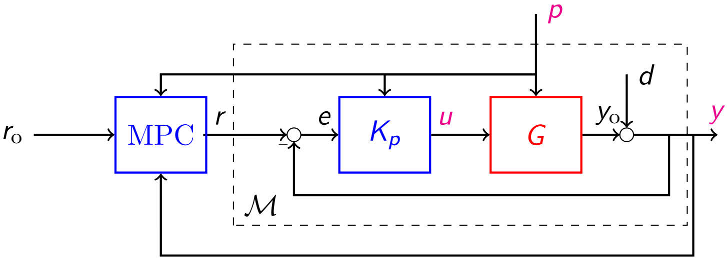
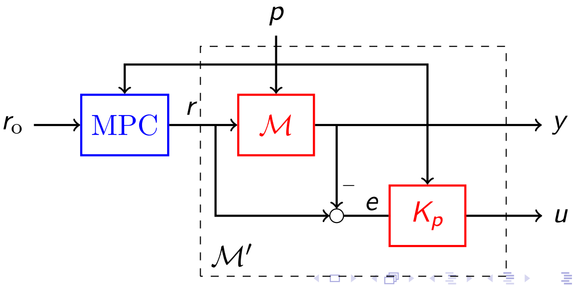
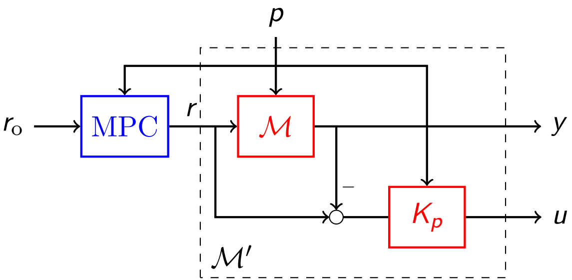


Control design scheme:

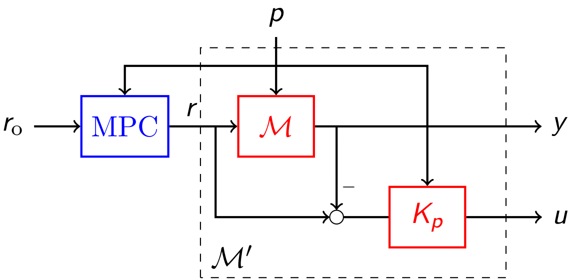


MPC design



$$\min_{\color{red}r} \sum_{t=1}^{N} Q_y \left(y(t+k|k) - r_o(t+k)\right)^2 +$$

MPC design



$$\min_{r} \sum_{t=1}^{N} Q_y \left(y(t+k|k) - r_{\mathrm{o}}(t+k)\right)^2 + \sum_{t=1}^{N_c} Q_r \left(r_{\mathrm{o}}(t+k) - r(t+k|k)\right)^2$$

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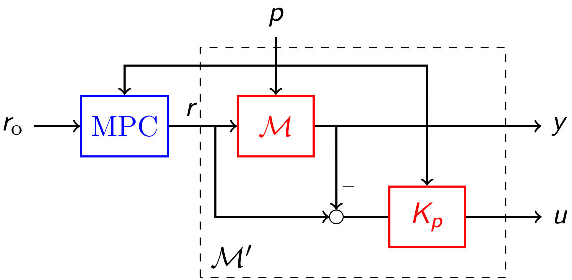

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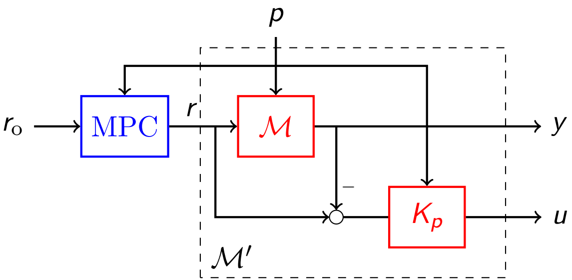

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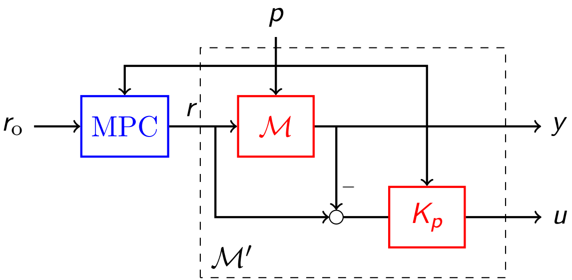

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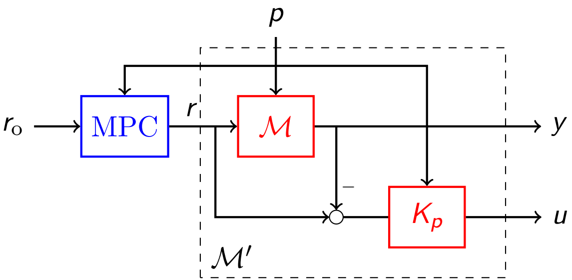

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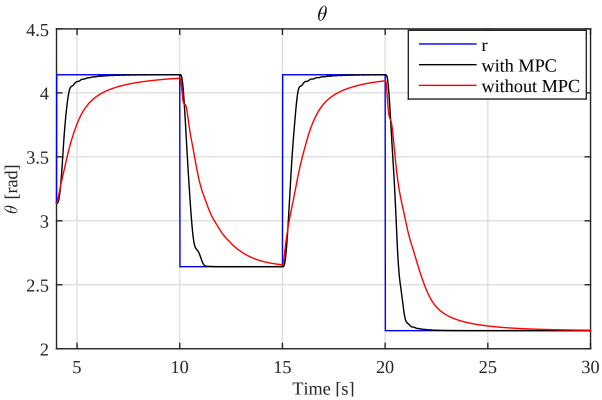
DC motor: obtained results



Average time required to compute the control law: 11 ms (MPC toolbox, Matlab)

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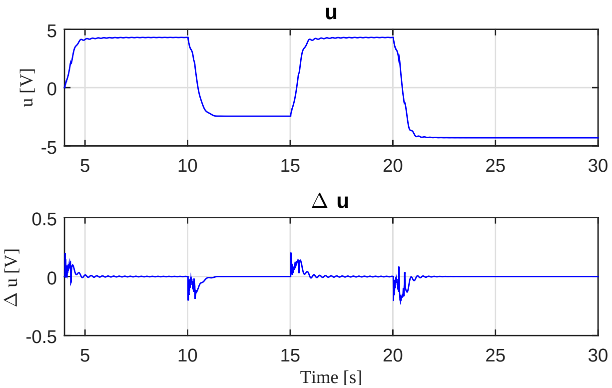

$|\Delta u| \leq 0.2$ V

Constraints on the input signal $|\Delta u| \leq 0.2$ V are fulfilled

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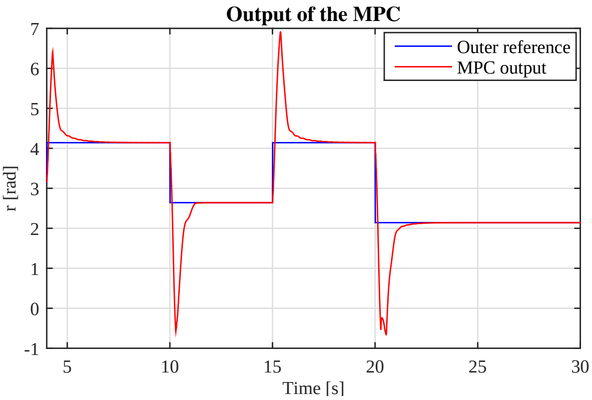

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- Real-world applications?