Inverse source estimation problems in magnetostatics

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Context

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- Planetary sciences, paleomagnetism, remanent magnetization of ancient rocks $\rightsquigarrow$ history and future of Earth magnetic field
- Magnetization not measurable
  $\rightsquigarrow$ measures of generated magnetic field
  $\rightsquigarrow$ inverse problems, non destructive inspection

How do rocks acquire magnetization?

- Igneous rocks (from Earth volcanoes, lava, magma; basalts)

- Thermoremanent magnetization, ferromagnetic particles (small magnets) follow the magnetic field:

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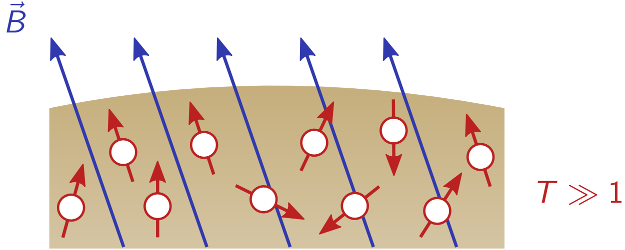

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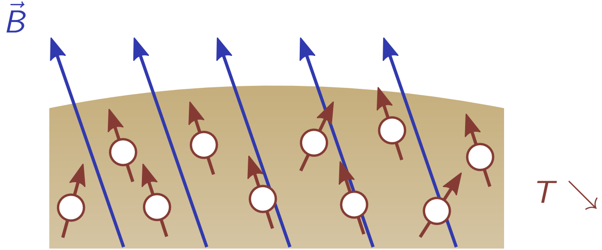

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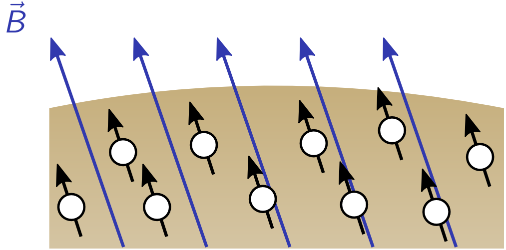

$T \simeq 1$

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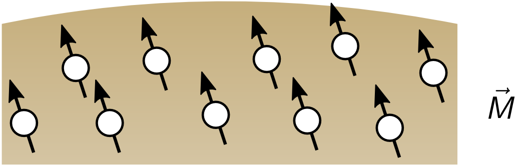

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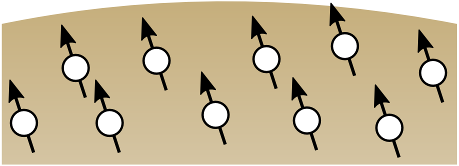


$\vec{M}$

- Can be subsequently altered, under high pressure or temperature

SQUID microscope

Scanning magnetic microscopes: (MIT, EAPS)
for weakly magnetized small samples

pedestal + sensor

sapphire window

$\rightsquigarrow$ map of the vertical component of the (tiny) magnetic field
on a rectangular region Q above the sample S

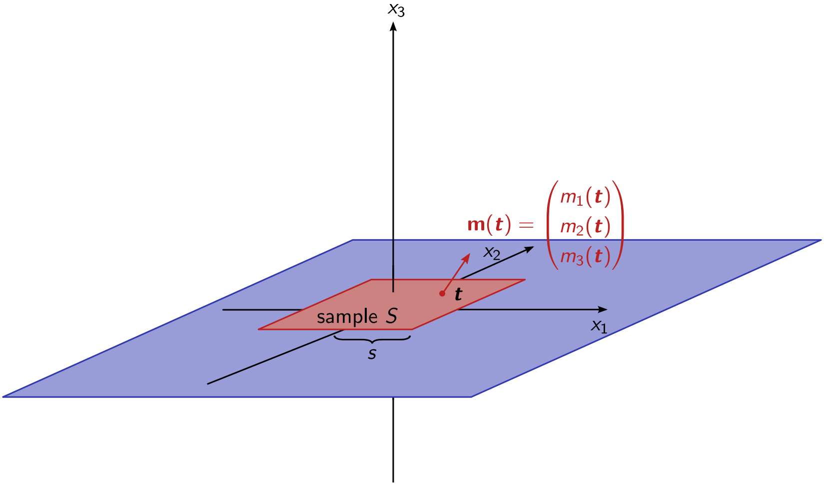
General scheme

Thin sample $\rightsquigarrow$ planar (rectangle) $S \subset \mathbb{R}^2$

support of unknown magnetization (source term) $\boldsymbol{m}$

$\vec{M} = \boldsymbol{m} = (m_1, m_2, m_3)^t$

B_3 on $Q \rightsquigarrow \boldsymbol{m}$ on S?



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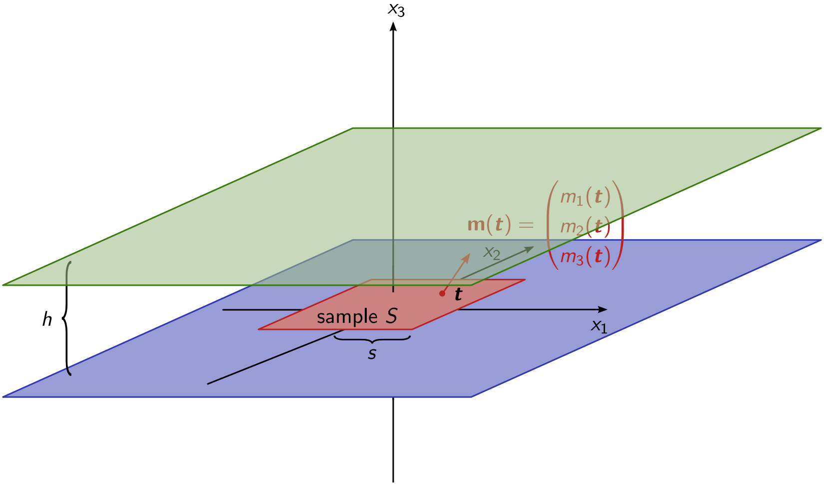

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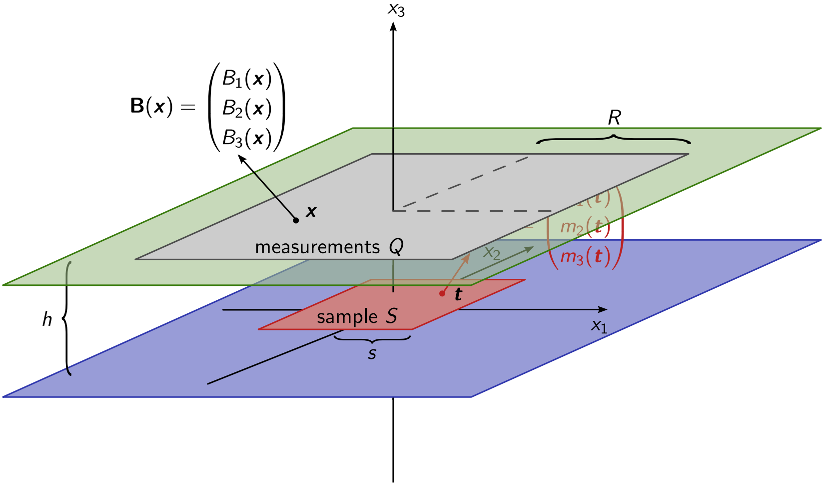
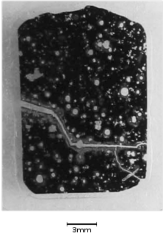
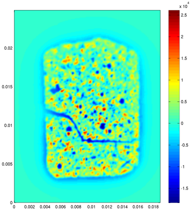
$$\boxed{B_3 \text{ on } Q \rightsquigarrow \boldsymbol{m} \text{ on } S?}$$



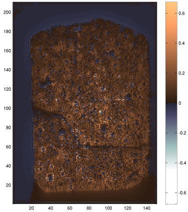
Hawaiian Basalt example $\rightsquigarrow$ magnetization recovery



picture
of thin sample



measures (nT)
of B_3 on $Q \times \{h\}$



estimated
magnetization (Am^2)
$\boldsymbol{m}$ on $S \times \{0\}$

knowing $\approx$ direction of $\boldsymbol{m}$ & after re-magnetization (MIT, EAPS)

not always feasible...

Problem setting, $B_3 = B_3[\boldsymbol{m}]$

- $\boldsymbol{m}$ generates a magnetic field at $\boldsymbol{x} \notin S$: $\boldsymbol{B}[\boldsymbol{m}](\boldsymbol{x}) = -\mu_0 \nabla U[\boldsymbol{m}](\boldsymbol{x})$

$$\text{where} \quad U[\boldsymbol{m}](\boldsymbol{x}) = \frac{1}{4\pi} \iint_S \frac{\langle \boldsymbol{m}(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y} \rangle}{|\boldsymbol{x} - \boldsymbol{y}|^3} \mathrm{d}\boldsymbol{y}$$

U scalar magnetic potential, μ_0 magnetic constant, ∇ 3D gradient

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- measurements $b_3[\boldsymbol{m}]$ of vertical component $B_3[\boldsymbol{m}] = -\mu_0 \partial_{x_3} U[\boldsymbol{m}]$ performed on square $Q \subset \mathbb{R}^2$ at height h (incomplete data)

∂_{x_i} partial derivative w.r.t. x_i

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- inverse problems: from pointwise values of $b_3[\boldsymbol{m}]$ on Q recover magnetization $\boldsymbol{m}$ or net moment $\langle \boldsymbol{m} \rangle$ on S

$$\langle \boldsymbol{m} \rangle = \begin{pmatrix} \langle m_1 \rangle \\ \langle m_2 \rangle \\ \langle m_3 \rangle \end{pmatrix}, \quad \langle m_i \rangle = \iint_S m_i(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}$$

average, $m_i \in L^2(S) \subset L^1(S)$, $i = 1, 2, 3$

Operators

Put $X = (x_1, x_2, x_3) \in \mathbb{R}^3$

$$\text{grad} \; = \nabla = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{pmatrix} \text{ with } \partial_{x_i} = \frac{\partial}{\partial_{x_i}}$$

$$\text{div} \; = \nabla \cdot \,, \quad \text{curl} \; = \nabla \times$$

$$\text{Laplacian} \; = \Delta = \text{div}\,(\text{grad}) = \nabla \cdot \nabla = \partial^2_{x_1^2} + \partial^2_{x_2^2} + \partial^2_{x_3^2}$$

Inverse recovery problems

Poisson-Laplace equation with source term in divergence form:

Maxwell, magnetostatics, time-harmonic, macroscopic

$$\begin{cases} \Delta\, U[\boldsymbol{m}] = \nabla \cdot \boldsymbol{m} \ \text{ in } \mathbb{R}^3 , \quad \text{supp } \boldsymbol{m} \subset S \times \{0\} \\ B_3 = B_3[\boldsymbol{m}] = \partial_{x_3} U[\boldsymbol{m}] \end{cases}$$

$U[\boldsymbol{m}]$ and $B_3[\boldsymbol{m}]$ harmonic functions in $\{x_3 > 0\}$

$$b_3[\boldsymbol{m}] = B_3[\boldsymbol{m}]_{|Q \times \{h\}} \text{ on } Q \times \{h\} \rightsquigarrow \langle \boldsymbol{m} \rangle \text{ or } \boldsymbol{m} \text{ on } S \times \{0\}\,?$$

$$\langle \boldsymbol{m} \rangle = \iint_S \boldsymbol{m}(\boldsymbol{y})\, d\,\boldsymbol{y} \in \mathbb{R}^3 , \quad \boldsymbol{m} \in L^2(S, \mathbb{R}^3) , \quad b_3[\boldsymbol{m}] \in L^2(Q)$$

Integral expression for $(x_1 , x_2) \in Q$ (also in terms of Poisson and Riesz transforms)

$$b_3\,[\boldsymbol{m}]\,(x_1 , x_2) = -\frac{\mu_0}{4\pi} \times \left(\partial_{x_3} \iint_S \frac{m_1(\boldsymbol{y})\,(x_1 - y_1) + m_2(\boldsymbol{y})\,(x_2 - y_2) + m_3(\boldsymbol{y})\, x_3}{((x_1 - y_1)^2 + (x_2 - y_2)^2 + x_3^2)^{3/2}}\, d\boldsymbol{y} \right)_{|x_3 = h} \qquad \boldsymbol{y} = (y_1 , y_2)$$

Inverse recovery problems

Data: $b_3\,[\boldsymbol{m}]$ on Q

- net moment $\langle m_i \rangle$ estimation:
  uniqueness, unstability $\rightsquigarrow$ regularization (BEP)

- preliminary step for magnetization recovery (non-uniqueness):
  $\rightsquigarrow$ mean values $\langle m_i \rangle$ of m_i on S $\rightsquigarrow$ $\langle \boldsymbol{m} \rangle$
  $\rightsquigarrow$ direction of $\boldsymbol{m}$

Inverse recovery problems

- Linear map:

$$b_3 : \begin{cases} L^2(S, \mathbb{R}^3) \to L^2(Q) \\ \boldsymbol{m} \mapsto B_3[\boldsymbol{m}]_{|Q\times\{h\}} = b_3[\boldsymbol{m}] \end{cases}$$

- Magnetization recovery (full inversion): recover $\boldsymbol{m}$ from $b_3[\boldsymbol{m}]$
  $\rightsquigarrow$ ill-posed (non uniqueness) because Ker $b_3 \neq \{0\}$, silent sources

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- Moment recovery: well-defined $i \in \{1, 2, 3\}$

$$\begin{cases} \text{Ran } b_3 \to \mathbb{R} \\ b_3[\boldsymbol{m}] \mapsto \langle m_i \rangle \end{cases}$$

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- Strategy: linear estimator $\forall \boldsymbol{m}$ of given norm, once and for all
find test functions $\phi_i \in L^2(Q)$ such that $\langle \phi_i,\ b_3[\boldsymbol{m}]\rangle_{L^2(Q)} \simeq \langle m_i \rangle$
$\rightsquigarrow$ best constrained approximation problems (BEP)

and asymptotic formulas, for large measurement area Q

Direct inversion of associated discrete problem: heavy, unstable

Strategy for net moment recovery

Determine $\phi_i \in L^2(Q)$ such that $\quad \langle b_3[\boldsymbol{m}], \phi_i \rangle_Q \simeq \langle m_i \rangle = \langle \boldsymbol{m}, \boldsymbol{e}_i \rangle_S$

$\langle \cdot, \cdot \rangle_{L^2(Q)}\,,\ \langle \cdot, \cdot \rangle_{L^2(Q,\mathbb{R}^2)} \rightsquigarrow \langle \cdot, \cdot \rangle_Q,\ \|\cdot\|_Q, \quad \langle \cdot, \cdot \rangle_S \rightsquigarrow \langle \cdot, \cdot \rangle_S,\ \|\cdot\|_S$

$\boldsymbol{e}_1 = (1,0,0)\,,\ \boldsymbol{e}_2 = (0,1,0)\,,\ \boldsymbol{e}_3 = (0,0,1)$ on S

$\langle b_3[\boldsymbol{m}], \phi_i \rangle_Q = \langle \boldsymbol{m}, b_3^*[\phi_i] \rangle_S$ with adjoint operator $b_3^* : L^2(Q) \to L^2(S, \mathbb{R}^3)$ to b_3

$\rightsquigarrow$ linear estimator for moment $\langle m_i \rangle$ — given bound on $\|\boldsymbol{m}\|_S$

$\rightsquigarrow$ trade-off between prescribed accuracy $\simeq$ and norm of ϕ_i

in Sobolev space, L^2 norm of gradient of ϕ

$\rightsquigarrow$ stability w.r.t. measurement errors, robustness

Tools for analysis

- Properties of Hilbert Sobolev spaces $W_0^{1,2}(\Omega) \subset L^2(\Omega)$ $\quad \Omega \subset \mathbb{R}^2$
- Integral expression of $b_3[\boldsymbol{m}]$ in terms of Poisson and Riesz transforms
- Properties of operator b_3 and its adjoint b_3^*
  - $\boldsymbol{e}_i \perp \operatorname{Ker} b_3$: silent sources possess vanishing net moments
    $\rightsquigarrow$ uniqueness of $\langle \boldsymbol{m} \rangle$ from $b_3\,[\boldsymbol{m}]$
  - $\operatorname{Ran} b_3$ dense in $L^2(Q)$

Net moment recovery, strategy

- no hope to find $\phi \in L^2(Q)$ such that

$$\langle b_3 [\boldsymbol{m}] , \phi \rangle_Q - \langle m_i \rangle = 0 \quad \forall \boldsymbol{m} \in L^2(S, \mathbb{R}^3)$$

because $\boldsymbol{e}_i \notin \text{Ran } b_3^*$ and $\langle b_3 [\boldsymbol{m}] , \phi \rangle_Q - \langle m_i \rangle = \langle \boldsymbol{m} , b_3^* [\phi] - \boldsymbol{e}_i \rangle_S$

- however, density result for $\boldsymbol{m} \in L^2(S, \mathbb{R}^3)$

$\exists \, \phi_n \in W_0^{1,2}(Q)$ s.t. $\quad |\langle b_3 [\boldsymbol{m}] , \phi_n \rangle_Q - \langle m_i \rangle| \to 0 \quad$ as $n \to \infty$

$$\inf_{\phi \in W_0^{1,2}(Q)} |\langle b_3 [\boldsymbol{m}] , \phi \rangle_Q - \langle m_i \rangle| = 0$$

- but it satisfies $\|\nabla_2 \, \phi_n\|_Q \to \infty$ $\quad \rightsquigarrow$ unstability

for $\boldsymbol{m} \in L^2(S, \mathbb{R}^3)$, we can find $\phi \in W_0^{1,2}(Q) \subset L^2(Q)$ s.t. $|\langle b_3 [\boldsymbol{m}] , \phi \rangle_Q - \langle m_i \rangle|$ arbitrarily small

at the expense of arbitrarily large $\|\nabla_2 \, \phi\|_Q$

$\rightsquigarrow$ needs for regularization, trade-off (error / constraint)

Bounded extremal problem

$$\left|\langle b_3\,[\boldsymbol{m}]\,,\,\phi\rangle_Q - \langle m_i\rangle\right| = \left|\langle \boldsymbol{m}\,,\,b_3^*\,[\phi] - \boldsymbol{e}_i\rangle_S\right| \leq \|\boldsymbol{m}\|_S\,\left\|b_3^*\,[\phi] - \boldsymbol{e}_i\right\|_S$$

Best constrained approximation, optimization issue $M > 0$

- $\exists$ unique solution $\phi_* \in W_0^{1,2}(Q)$ to (BEP)

$$\left\|b_3^*\,[\phi_*] - \boldsymbol{e}_i\right\|_S = \min_{\phi\in W_0^{1,2}(Q),\,\|\nabla_2\,\phi\|_Q\leq M} \left\|b_3^*\,[\phi] - \boldsymbol{e}_i\right\|_S$$

and $\|\nabla_2\,\phi_*\|_Q = M$

- it satisfies the critical point equation (CPE) $\rightsquigarrow$ Tykhonov type regularization

$$b_3\,b_3^*\,[\phi_*] - \lambda\,\Delta_2\,\phi_* = b_3\,[\boldsymbol{e}_i] \text{ on } Q$$

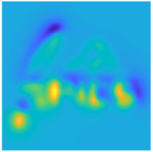
for unique $\lambda > 0$ s.t. $\|\nabla_2\,\phi_*\|_Q = M$ also, $\phi_* \in C^{1/2}(\bar{Q})$

Lagrange parameter $\lambda \to 0$ iff constraint $M \to \infty$ & criterion error $\to 0 \rightsquigarrow$ resolution algorithm: iterate in λ

Ongoing, solving (CPE), preliminary numerics

$\boldsymbol{m} = (m_1, m_2, m_3)$ on S

$b_3[\boldsymbol{m}]$ on Q



synthetic data (matlab)

simulated $\boldsymbol{m}$, $b_3[\boldsymbol{m}]$

$R = 2.55$, $s = 1.97$, $h = 0.27$ mm, $|Q|/|S| \simeq 1.3$, $|S|/h \simeq 14.6$

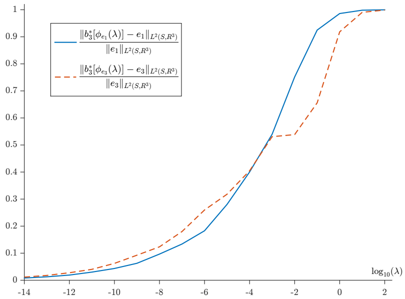
100×100 overlapping squares, finite elements $\mathbb{P}_0$ on S $\mathbb{Q}_1$ on Q, for $W_0^{1,2}(Q)$ basis functions ψ

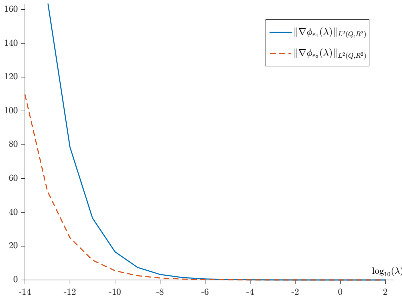
(CPE) Dirichlet problems for elliptic 2D PDE: $\phi_* \in W_0^{1,2}(Q)$, $\phi_{*|\partial Q} = 0$

$$\langle b_3^* [\phi_*] , b_3^* [\psi] \rangle_S + \lambda \langle \nabla_2 \phi_* , \nabla_2 \psi \rangle_Q = \langle b_3 [\boldsymbol{e}_i] , \psi \rangle_Q$$

Solving (CPE), preliminary numerics

⇝ towards a numerical magnetometer, $\phi_* = \phi_{\boldsymbol{e}_i}$, $i = 1, 2, 3$





criterion error (cost) on S

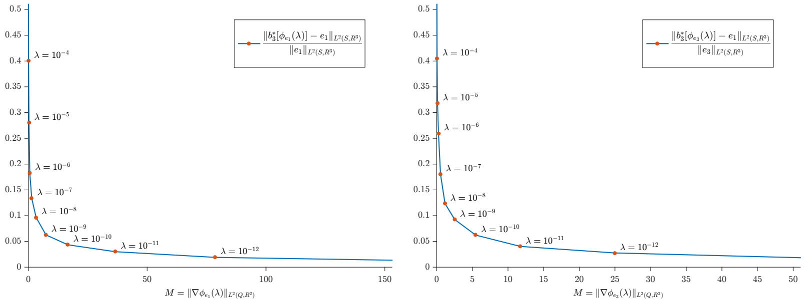
constraint M on Q

$\| b_3^* [\phi_{\boldsymbol{e}_i}] - \boldsymbol{e}_i \|_S$

$\| \nabla_2 \phi_{\boldsymbol{e}_i} \|_Q$

for $i = 1, 3$, w.r.t. $\log_{10}(\lambda)$

Solving (CPE), preliminary numerics



L-curves, $i = 1$ (left), $i = 3$ (right):

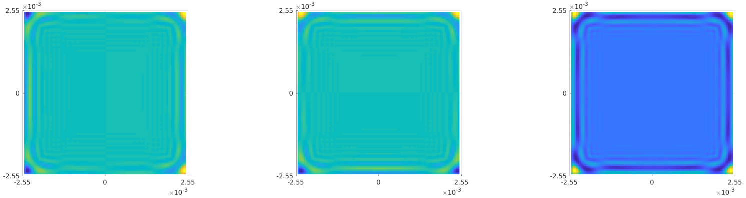
criterion error on S w.r.t. constraint M on Q, at values of λ

$$\left\| b_3^* \left[\phi_{\boldsymbol{e}_i}\right] - \boldsymbol{e}_i \right\|_S \text{ w.r.t. } \left\| \nabla_2 \, \phi_{\boldsymbol{e}_i} \right\|_Q$$

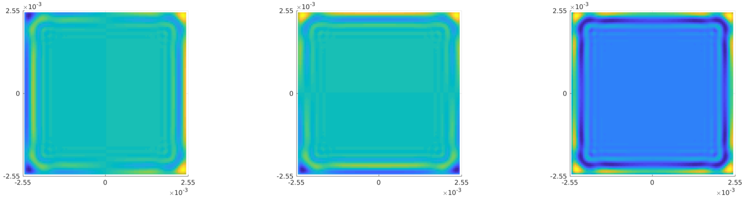
$\rightsquigarrow$ trade-off between error and constraint

Solving (CPE), preliminary numerics

$\phi_{e_i}(\lambda_1)$ on Q, for $i = 1, 2, 3$ $\lambda_1 = 10^{-11}$

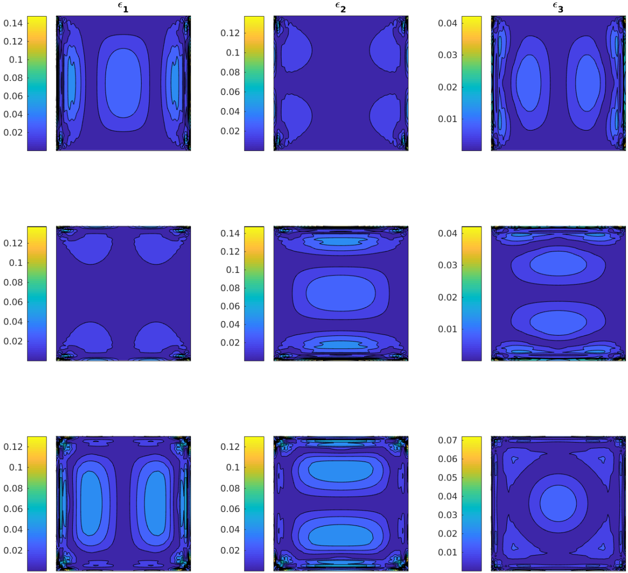


$\phi_{e_i}(\lambda_2)$ on Q, for $i = 1, 2, 3$ $\lambda_2 = 10^{-8}$



Solving (CPE), preliminary numerics

errors ϵ_k on component $k = 1, 2, 3$ of $b_3^* \left[\phi_{\boldsymbol{e}_i}(\lambda_1) \right] - \boldsymbol{e}_i$ on S for $i = 1, 2, 3$ (rows) $\qquad \lambda_1 = 10^{-11}$



Solving (CPE), preliminary numerics

1st row: actual values of components $\langle m_i \rangle$, $i = 1, 2, 3$

	-7.4059e-05	-1.1218e-04	4.0880e-05	
$\lambda = 10^{-6}$	-6.4500e-05	-9.4439e-05	3.9343e-05	14.38%
10^{-7}	-6.7673e-05	-1.0144e-04	3.9859e-05	8.92%
10^{-8}	-6.9700e-05	-1.0586e-04	4.0132e-05	5.49%
10^{-9}	-7.1047e-05	-1.0861e-04	4.0319e-05	3.35%
10^{-10}	-7.2020e-05	-1.1031e-04	4.0461e-05	1.99%
10^{-11}	-7.2749e-05	-1.1128e-04	4.0578e-05	1.15%
10^{-12}	-7.3267e-05	-1.1180e-04	4.0664e-05	0.64%

next rows: estimated values values $\langle b_3[\boldsymbol{m}]\,,\, \phi_{\boldsymbol{e}_i}[\lambda] \rangle_Q$

last column: relative errors on moment $\langle \boldsymbol{m} \rangle$

Ongoing, next

- Resolution schemes, numerical analysis (ctn), test noisy & actual data & other resolution schemes:
  - iterative, $\phi_n \to \phi_* = \phi_{\boldsymbol{e}_i}$ $\quad n \to \infty$

    $-\lambda\, \varrho\, \Delta_2\, \phi_n + \phi_n = -\varrho\, b_3\, b_3^* \, [\phi_{n-1}] + \phi_{n-1} + \varrho\, b_3\, [\boldsymbol{e}_i]$ and $\phi_{n|\partial Q} = 0$, $n \geq 1$
  - in Fourier basis — product of sin of separated variables
  - asymptotic estimations — $|Q| > |S|$ or $R > s$
- Local moments determined by $b_3[\boldsymbol{m}]$ if $\boldsymbol{m}$ of minimal $L^2(S)$ norm
- (BEP) for general $\boldsymbol{e} \in \overline{\text{Ran } b_3^*} \setminus \text{Ran } b_3^* \rightsquigarrow$ higher order moments
- Use $\langle \boldsymbol{m} \rangle$ for magnetization $\boldsymbol{m}$ estimation

References

• L. Baratchart, D. Hardin, E.A. Lima, E.B. Saff, B.P. Weiss, Characterizing kernels of operators related to thin-plate magnetizations via generalizations of Hodge decompositions, Inverse Problems, 2013

• L. Baratchart, S. Chevillard, J. Leblond, Silent and equivalent magnetic distributions on thin plates, Theta Series in Advanced Mathematics, to appear, hal-01286117

Also, in planetary sciences, paleomagnetism:

• magnetizations $\boldsymbol{m}$ in $L^1(S, \mathbb{R}^3)$ or distributions

• for 3D samples S, pointwise dipolar sources chondrules, also Moon rocks, ...

In brain imaging (medical engineering), electroencephalography (EEG), source and conductivity inverse problems,

spherical geometry, software FindSources3D (+ MEG)