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Are right-half plane zeros necessary for inverse response? It depends . . .

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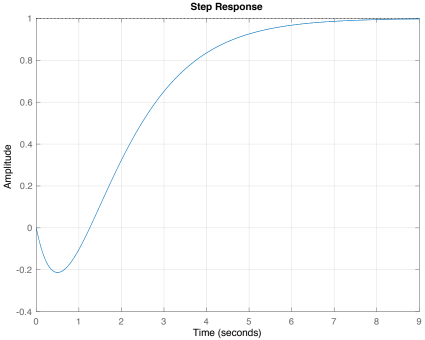
Inverse response

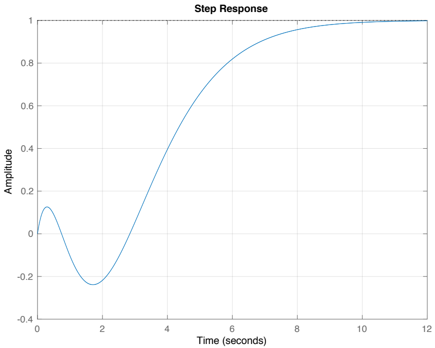
What is it?



$$\frac{1-s}{(1+s)^2}$$

Inverse response

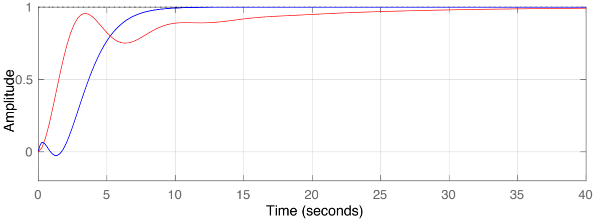
What is it?



$$\frac{(1-s)^2}{(1+s)^3}$$

Inverse response

What is it?



Don't confuse with *undershoot*

Inverse response

Where does it occur?

- Aircraft — pitch to climb.
- Process control — increase water flow to drum boiler.
- Neuroscience — Hodgkin-Huxley voltage clamp (nerve axon).
- Economics — devalue currency to improve balance of payments.

What is already known?

Odd number of zeros in RHP

We consider LTI SISO systems only.
Let $y_s(t)$ be step response of system $G(s)$.

$$G(s) = \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}, \quad m < n, \quad \Re(p_i) < 0,\ z_i \neq 0,\ \forall i$$

$$y_s(0) = 0 \qquad \text{and} \qquad y_s(\infty) = \frac{(-z_1)(-z_2)\cdots(-z_m)}{(-p_1)(-p_2)\cdots(-p_n)}$$

$-p_i > 0$ so $y_s(\infty) < 0$ if and only if there is an odd number of zeros in the RHP.
From Initial Value Theorem:

$$\left.\frac{d^k y_s(t)}{dt^k}\right|_{t=0} = 0 \text{ for } k < n-m, \quad \text{and} \quad \left.\frac{d^{n-m} y_s(t)}{dt^{n-m}}\right|_{t=0} = 1$$

What is already known?

Odd number of zeros in RHP

- So $y_s(t) > 0$ for small enough t.
- But $y_s(\infty) < 0$ iff there is an odd number of zeros in the RHP.
- So, **if** you define *inverse response* to be

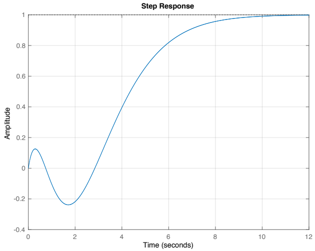
$$\operatorname{sign} y_s(\varepsilon) = -\operatorname{sign} y_s(\infty) \text{ for small enough } \varepsilon$$

  then at least one zero in RHP is *necessary*
  and an odd number of zeros in RHP is *sufficient*.

  (Norimatsu and Ito 1961, Vidyasagar 1986).
- This result extends(?) to irrational transfer functions
  (El-Khoury *et.al.* 1993, Widder 1934) — needs different approach.

What is already known?

Positive real zeros



- Define inverse response as: $\exists t_1, t_2 : \mathrm{sign} y_s(t) = -\mathrm{sign} y_s(\infty) \quad \text{for} \quad t \in [t_1, t_2]$
- $G(z) = 0$ for positive real z is *sufficient* for inverse response.
- **Proof:**

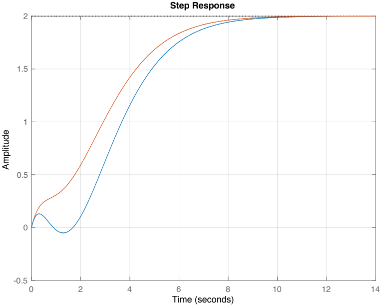
$$0 = \frac{G(z)}{z} = \int_0^\infty y_s(t) e^{-zt} dt \quad \text{and} \quad e^{-zt} > 0.$$

(Stewart and Davison 2006)

What is already known?

Complex zeros in the RHP are **not** *sufficient for inverse response*

Complex zeros in RHP are not sufficient



Real zeros in RHP are not necessary

$$\frac{s^2 - 0.2s + 2}{(s+1)^3}$$

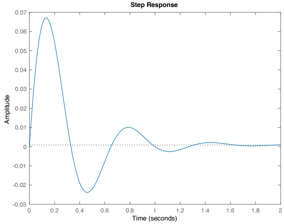
$$\frac{s^2 - 2s + 2}{(s+1)^3}$$

Zeros at $0.1 \pm 1.41i$

Zeros at $1 \pm i$

Are RHP zeros necessary for inverse response?

Short answer: **No**

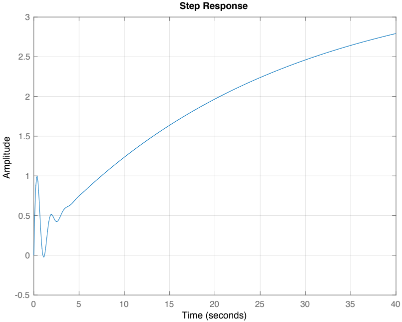


$$\frac{s + 0.08707}{s^2 + 6.041s + 101.4}$$

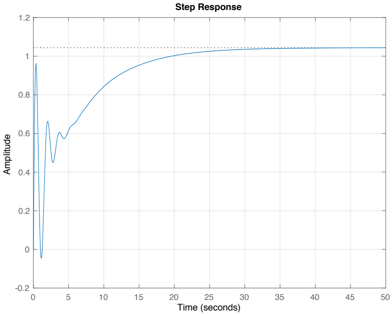
- Final value $= 8.6 \times 10^{-4} > 0$.
- Not what we mean by 'inverse response'.
- Define ρ-*inverse response* as: $\mathrm{sign}\, y_s(t) = -\mathrm{sign}\, y_s(\infty),\ \ t \in [t_1, t_2]$ **and** $\frac{|y_s(\infty)|}{\sup_{t>0} |y_s(t)|} \geq \rho$, $(0 < \rho \leq 1)$.
- **Claim:** 'Inverse response' really means 'ρ-inverse response with ρ "close to" 1'.

Are RHP zeros necessary for 1-inverse response?

No



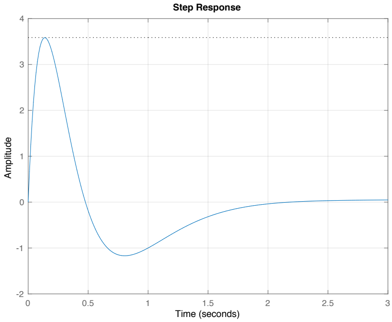
Poles: -0.0395 and $-1.2200 \pm 4.0901i$.
Zeros: $-0.2964 \pm 0.6064i$.

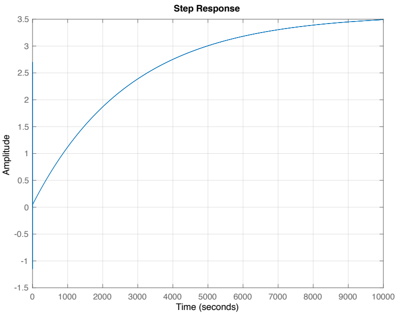


Poles: $-0.9385 \pm 3.7971i$, $-0.5400 \pm 0.6116i$ and -0.1584.
Zeros: $-0.3486 \pm 1.2712i$ and $-0.2262 \pm 0.3714i$.

RHP zeros necessary for 1-inverse response if real poles only?

No: 4th order system



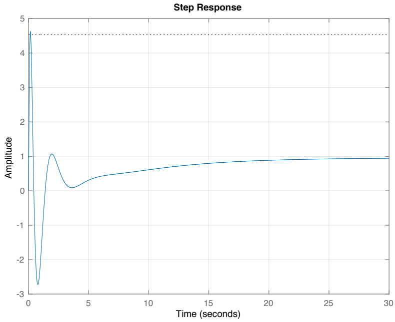


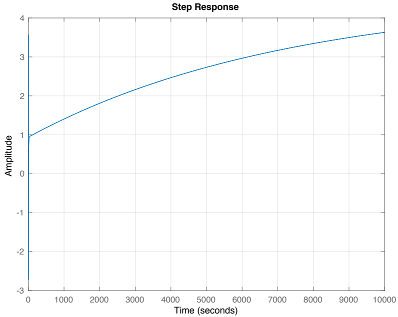
Poles: $\boxed{-4.3453, -4.2058, -4.1918,}\ -0.0004.$

Zeros: $\boxed{-0.0072 \pm 0.2343i,}\ -0.0275.$

RHP zeros necessary for 1-inverse response if real poles only?

No: 9th order system





Poles: $\boxed{-420.9554,}$ $-3.0211, -2.7254, -1.8889, -1.6949, -1.4678,$
$-0.6926, -0.1686,$ $\boxed{-0.0001.}$

Zeros: $\boxed{-8.4498 \times 10^5, -6.6269 \times 10^4,}$ $-1.5655 \times 10^3 \pm 81.032i,$
$-478.39 \pm 58.595i, -14.043 \pm 46.966i.$

Searching for counter-examples

Optimise over bounded domain

- Search over interior of unit disk.
  - Use Levinson-Durbin parametrisation of Schur polynomials.
  - Use root positions to parametrise Schur polynomials with real roots.
- Use bilinear mapping from $\{z : |z| < 1\}$ to $\{s : \Re(s) < 0\}$.
  - $s \leftrightarrow \frac{z-1}{z+1}$ or $z \leftrightarrow \frac{1+s}{1-s}$
  - Maps from Schur to Hurwitz polynomials.
  - Linear mapping between polynomial coefficients.
- Optimisation: minimise $|y_s(1) - \alpha|$
  - $\alpha = 0$ or $\alpha = -1$ or ...
  - Can choose $t = 1$ wlog, because $f(t/\beta) \leftrightarrow \beta F(\beta s)$
  - Note that $y_s(\infty) > 0$ since we allow only LHP poles and zeros.

Searching for counter-examples

Levinson-Durbin algorithm

- 1-1 mapping between Schur polynomials and *reflection coefficients*. (Think of polynomial as AR process.)
- $p(z) = p_0 z^k + p_1 z^{k-1} + \cdots + p_k$, $\mathbf{p} = [p_0, p_1, \ldots, p_k]^T$
  $\bar{p}(z) = p_k z^k + p_{k-1} z^{k-1} + \cdots p_0$, $\bar{\mathbf{p}} = [p_k, p_{k-1}, \ldots, p_0]^T$.
- $\mathbf{p} = 1$; for $k = 1 : n$,
$$\mathbf{p} = \begin{bmatrix} \mathbf{p} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ r(k)\bar{\mathbf{p}} \end{bmatrix}$$
  end
- *Theorem:* $|r(k)| < 1$ and $r(k) \in \mathbb{R}$ for $1 \leq k \leq n$ gives monic Schur $p(z)$ of degree n.
  To every such $p(z)$ there corresponds a *unique* set $\{r(k) : |r(k)| < 1,\ r(k) \in \mathbb{R},\ k = 1, \ldots, n\}$.

1. Start with Schur polynomial $p(z) = p_0 z^n + p_1 z^{n-1} + \cdots + p_n$.
2. Let $\dfrac{q(s)}{(1-s)^n} = p\left(\dfrac{1+s}{1-s}\right)$ and $q(s) = q_0 s^n + q_1 s^{n-1} + \cdots + q_n$.
3. Then $q(s)$ is Hurwitz and $\mathbf{q} = P\mathbf{p}$, where

$$\mathbf{p} = [p_0, p_1, \ldots, p_n]^T, \quad \mathbf{q} = [q_0, q_1, \ldots, q_n]^T$$

$$P(1,j) = 1, \quad P(i, n+1) = (-1)^{i-1}\binom{n}{i}$$

$$P(i,j) = P(i-1,j) + P(i-1,j+1) + P(i,j+1) \text{ for } i > 1, j < n+1.$$

4. Useful fact (but not needed here): $P^{-1} = 2^{-n}P$.

Searching for counter-examples

Optimisation algorithm

- `fmincon` with default options (= Interior Point)
  - Don't need global optimum.
  - Start from *MAXRAND* initial guesses.
  - Up to *MAXITER* steps from each initial guess.
- Search gets *easier* as order is increased
  - eg $n = 3$: failure with $MAXRAND = 40$ and $MAXITER = 100$,
  - but $n = 9$: only needs 2 initial guesses and 10–20 steps.
- No need to normalise (eg $q_0 = 1$) or to remove pole-zero cancellations.

Conclusions

- RHP zeros are *not* necessary for inverse response.
- But excluding them can lead to strange behaviours
  - Time scale separation
  - Initial transient often reaches final value.
  - Implications for feedback design?
- If you want a low-order black-box model then allowing RHP zeros makes life easier — especially with only real poles.
- Real RHP zeros are sufficient for inverse response.
- If the initial response is in the "wrong" direction then real RHP zeros *are* necessary.